A superspace embedding of the Wess-Zumino model

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Abstract

We embed the Wess-Zumino (WZ) model in a wider superspace than the one described by chiral and anti-chiral superfields.

1. There is a systematic and interesting formalism for embedding, developed by Batalin, Fradkin, Fradkina, and Tyutin (BFFT) [1], where theories with second-class constraints [2] are transformed into more general (gauge) theories where all constraints become first-class. The transformation of constraints from second to first-class is achieved after extending the phase space by means of auxiliary variables under the general rule that there is one pair of canonical variables for each second class constraint. The method is iterative and can stop in the first step [3] or can go on indefinitely [4,5]. In any case, after all constraints have been transformed into first-class, it is necessary to look for the Hamiltonian corresponding to this new theory. The method also permit us to obtain any involutive quantity that has zero Poisson brackets with all the constraints. The embedding Hamiltonian can be obtained in this way, starting from the initial canonical Hamiltonian and iteratively calculating the corresponding corrections.

There is another manner to obtain an embedding Hamiltonian, which consists in using the BFFT method to obtain involutive coordinates [4]. The canonical Hamiltonian is then rewritten in terms of these new coordinates that automatically give it the involutive condition. Of course, the embedding Hamiltonians obtained from these two different ways are not necessarily equal. This means that for some specific theory there may exist more than one possible embeddings. It is also opportune to say, on the other hand, that there are theories which cannot be embedded [6].

One of the interesting problem that the BFFT method could be addressed is the covariant quantization of superparticles and superstrings, that remained opened for a long time. This problem has been solved in a embedding procedure but differently of the BFFT method [7]. In fact, the meaning of embedding in field theory can be taken as much wider than the cases described by the BFFT method. The important point is that the embedding theory contains all the physics of the embedded one. We mention, for example, even the general procedure of supersymmetrization is an example of embedding.

We would like to address the present paper to this point of view of considering the embedding procedure in a wider way. We concentrate on the WZ model [8] in superfield language [9,10]. Conventionally, the WZ model is always developed in terms of chiral and antichiral superfields, that are examples of irreducible superfields. We shall consider here a kind of embedding where we describe the WZ model by using a more general superfield representation. We shall see that, contrarily to the bosonic nature of the chiral and antichiral superfields, the general superfield we have to use is fermionic. We shall also see that there are two possible terms that can figure in the Lagrangian and having a relative parameter between them. The consistency of the obtained theory can be verified by showing it has the same physics of the WZ model. Finally, we know that a characteristic of embedding theories is that they have more symmetries than the embedded one. The same also occurs here. We can show that for an specific value of the relative parameter between the two terms of the Lagrangian, there is a kind of gauge symmetry relating all the fields of the theory.

2. In order to fix the notation and make future comparisons, let us write down the general form of the real and scalar superfield,

$$\begin{aligned}\Phi(x,\theta) = A(x) + \bar\theta\psi(x) + \frac{1}{2}\,\bar\theta\theta\,B(x)\\ +\frac{i}{2}\,\bar\theta\gamma_5\theta\,C(x) + \frac{1}{2}\,\bar\theta\gamma^\mu\gamma_5\theta\,A_\mu(x)\\ +\frac{1}{2}\,\bar\theta\theta\bar\theta\,\lambda(x) + \frac{1}{4}\,(\bar\theta\theta)^2\,D(x)\,,\end{aligned} \tag{1}$$

Here, all the spinors are Majorana and are in the Majorana representation (their components are real). We observe that it contains eight bosonic and eight fermionic degrees of freedom. We are going to work in four component notation for the spinor fields. In the Appendix, we

give more details about the notation and convention we are using and list some useful identities.

The irreducible positive and negative chiral superfields, that contains just four components, are given by

$$\begin{aligned}\Phi_+(x,\theta) &= \phi(x) + \frac{i}{2}\,\bar\theta\gamma^\mu\gamma_5\theta\partial_\mu\phi - \frac{1}{8}\,(\bar\theta\theta)^2\,\Box\phi \\ &+\frac{1}{2}\,\bar\theta(1+\gamma_5)\,\psi(x) - \frac{i}{4}\,\bar\theta\theta\bar\theta\gamma^\mu(1+\gamma_5)\,\partial_\mu\psi \\ &+\frac{1}{4}\,\bar\theta(1+\gamma_5)\theta\,F(x)\,, \end{aligned} \tag{2}$$

$$\begin{aligned}\Phi_-(x,\theta) &= \phi^*(x) - \frac{i}{2}\,\bar\theta\gamma^\mu\gamma_5\theta\partial_\mu\phi^* - \frac{1}{8}\,(\bar\theta\theta)^2\,\Box\phi^* \\ &+\frac{1}{2}\,\bar\theta(1-\gamma_5)\,\psi(x) - \frac{i}{4}\,\bar\theta\theta\bar\theta\gamma^\mu(1-\gamma_5)\,\partial_\mu\psi \\ &+\frac{1}{4}\,\bar\theta(1-\gamma_5)\theta\,F^*(x)\,. \end{aligned} \tag{3}$$

The WZ model [8,10] is directly obtained (up to some general constant factor) from an action given by the product of positive and negative chiral superfields, $S = \int d^4x d^4\theta\,\Phi_+\Phi_-$.

3. We first observe that the formulation of a supersymmetric theory, using general superfields and that contains the WZ model as a particular case, cannot be done in terms of covariant derivatives over the scalar superfield. This is so because it would violate the correct mass dimension of the superfield Lagrangian, that should be two. The correct way is starting from a fermionic superfield, whose general form reads

$$\begin{aligned}\Psi_\alpha(x,\theta) &= \chi_\alpha(x) + \theta_\alpha\phi(x) + \frac{1}{2}\,\bar\theta\theta\,\psi_\alpha(x) \\ &+\frac{i}{2}\,\bar\theta\gamma_5\theta\,\lambda_\alpha(x) + \frac{1}{2}\,\bar\theta\gamma^\mu\gamma_5\theta\,\psi_{\mu\alpha}(x) \\ &+\frac{1}{2}\,\bar\theta\theta\theta_\alpha F(x) + \frac{1}{4}\,(\bar\theta\theta)^2\,\eta_\alpha(x)\,. \end{aligned} \tag{4}$$

Consequently, the form of $\bar\Psi_\alpha(x,\theta)$ reads

$$\begin{aligned}\bar\Psi_\alpha(x,\theta) &= \bar\chi_\alpha(x) + \bar\theta_\alpha\phi^*(x) + \frac{1}{2}\,\bar\theta\theta\,\bar\psi_\alpha(x) \\ &+\frac{i}{2}\,\bar\theta\gamma_5\theta\,\bar\lambda_\alpha(x) + \frac{1}{2}\,\bar\theta\gamma^\mu\gamma_5\theta\,\bar\psi_{\mu\alpha}(x) \\ &+\frac{1}{2}\,\bar\theta\theta\bar\theta_\alpha F^*(x) + \frac{1}{4}\,(\bar\theta\theta)^2\,\bar\eta_\alpha(x)\,. \end{aligned} \tag{5}$$

If we consider the fermionic superfield with mass dimension $\frac{1}{2}$, the mass dimension of the component fields are

$$\begin{aligned}&[\chi] = \frac{1}{2}\,, \qquad [\phi] = 1\,, \qquad [\psi] = [\lambda] = [\psi_\mu] = \frac{3}{2}\,, \\ &[F] = 2\,, \qquad [\eta] = \frac{5}{2}\,. \end{aligned} \tag{6}$$

Notice that actually ϕ, ψ, and F have the same mass dimensions of the corresponding fields of the WZ model.

The supersymmetry transformations of the component fields can be directly obtained by the general supersymmetry transformation relation

$$\delta\Psi_\alpha = (\bar\xi Q)\,\Psi_\alpha\,, \tag{7}$$

which leads to

$$\begin{aligned}\delta\chi_\alpha &= \xi_\alpha\phi\,, \\ \delta\phi &= -\frac{i}{4}\bar\xi\partial\!\!\!/\chi - \frac{1}{4}\bar\xi\psi - \frac{i}{4}\bar\xi\gamma_5\lambda + \frac{1}{4}\bar\xi\gamma_5\gamma^\mu\psi_\mu\,, \\ \delta\psi_\alpha &= \frac{i}{2}\big(\partial\!\!\!/\phi\xi\big)_\alpha + \frac{1}{2}\xi_\alpha F\,, \\ \delta\lambda_\alpha &= \frac{1}{2}\big(\gamma_5\partial\!\!\!/\phi\xi\big)_\alpha + \frac{i}{2}\big(\gamma_5\xi\big)_\alpha F\,, \\ \delta\psi_{\mu\alpha} &= \frac{i}{2}\big(\gamma_5\gamma_\mu\partial\!\!\!/\phi\xi\big)_\alpha - \frac{1}{2}\big(\gamma_5\gamma_\mu\xi\big)_\alpha F\,, \\ \delta F &= \frac{i}{4}\bar\xi\partial\!\!\!/\psi + \frac{1}{4}\bar\xi\gamma_5\partial\!\!\!/\lambda + \frac{i}{4}\bar\xi\gamma^\mu\gamma^\nu\gamma_5\partial_\mu\psi_\nu - \frac{1}{4}\bar\xi\eta\,, \\ \delta\eta_\alpha &= \frac{i}{2}\Big(\partial\!\!\!/F\xi\Big)_\alpha\,. \end{aligned} \tag{8}$$

We observe that the usual transformations of the component fields of the WZ model are embodied in (8).

Before going on, it is opportune to make a comment about the number of bosonic and fermionic degrees of freedom that appear in (4) and (5). At first sight, they are not the same. There are thirty two fermionic degrees of freedom and apparently much less bosonic ones. What happens is that the bosonic quantities ϕ and F are not representing just single fields. Instead the quantity $\theta_\alpha\phi$, we must read more generically [11]

$$\begin{aligned}\theta_\alpha\phi \longrightarrow\; &\theta_\alpha\phi + (\gamma_5\theta)_\alpha\tilde\phi + (\gamma^\mu\theta)_\alpha A_\mu \\ &+ (\gamma_5\gamma^\mu\theta)_\alpha\tilde A_\mu + \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha B_{\mu\nu} \end{aligned} \tag{9}$$

that corresponds to sixteen degrees of freedom and the same occurs for the term with F.

4. The most general supersymmetric Lagrangian density expressed in terms of the spinor superfields, not containing high derivatives and nonlocal terms, has the form (up to some overall constant factor)

$$\mathcal{L} = \bar D_\alpha\Psi_\alpha D_\beta\bar\Psi_\beta + a\,\bar D_\alpha\Psi_\beta D_\alpha\bar\Psi_\beta\,, \tag{10}$$

where a is a relative normalization parameter that shall be conveniently fixed. After a long algebraic calculation, we obtain that the $(\bar\theta\theta)^2$ component of $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}_{(\bar\theta\theta)^2} &= \Big(2a+\frac{1}{2}\Big)\bar\psi\eta - \Big(2a+\frac{1}{2}\Big)\bar\lambda\partial^\mu\psi_\mu \\ &-\Big(a+\frac{1}{4}\Big)\bar\psi\Box\chi + (a-1)FF^* - a\phi\Box\phi^* \\ &+\frac{i}{4}\bar\psi\partial\!\!\!/\psi + \frac{1}{2}\bar\psi\gamma_5\partial\!\!\!/\lambda + \frac{i}{2}\bar\psi\gamma_5\partial^\mu\psi_\mu \\ &+\frac{i}{2}\bar\lambda\gamma_5\eta + \frac{i}{4}\bar\lambda\partial\!\!\!/\lambda + \frac{1}{2}\bar\psi_\mu\gamma^\mu\gamma_5\eta \end{aligned}$$

$$+\frac{i}{4}\bar\psi_\mu\gamma^\mu\gamma^\nu\partial\!\!\!/\psi_\nu+\frac{i}{2}\bar\eta\partial\!\!\!/\chi-\frac{i}{4}\bar\chi\gamma_5\Box\lambda$$
$$+\frac{1}{4}\bar\chi\gamma_5\gamma^\mu\gamma^\nu\partial\!\!\!/\partial_\mu\psi_\nu\,. \tag{11}$$

We notice that the bosonic quantities ϕ and F do not mix with any fermionic fields and their equations of motion are the usual ones that appear in the WZ model (up to the generic scaling parameter a). Concerning the equations of motion for the fermionic fields, we have

$$2(4a+1)\,\eta-(4a+1)\Box\chi+2i\partial\!\!\!/\psi+2\gamma_5\partial\!\!\!/\lambda+2i\gamma_5\partial^\mu\psi_\mu=0\,, \tag{12}$$

$$(4a+1)\,\psi+i\gamma_5\lambda-\gamma_5\gamma^\mu\psi_\mu+i\partial\!\!\!/\chi=0\,, \tag{13}$$

$$2(4a+1)\,\partial^\mu\psi_\mu+2\gamma_5\partial\!\!\!/\psi-2i\gamma_5\eta-2i\partial\!\!\!/\lambda+i\gamma_5\Box\chi=0\,, \tag{14}$$

$$2(4a+1)\,\partial^\mu\lambda-2i\gamma_5\partial^\mu\psi+2\gamma^\mu\gamma_5\eta+i(\gamma^\mu\gamma^\rho\gamma^\nu+\gamma^\nu\gamma^\mu\gamma^\rho)\partial_\nu\psi_\rho+\gamma_5\gamma^\nu\gamma^\mu\partial\!\!\!/\partial_\nu\chi=0\,, \tag{15}$$

$$(4a+1)\,\Box\psi-2i\partial\!\!\!/\eta+i\gamma_5\Box\lambda-\gamma_5\gamma^\mu\gamma^\nu\partial\!\!\!/\partial_\mu\psi_\nu=0\,. \tag{16}$$

As it has been emphasized, the procedure of embedding must not affect the physics we already know for the initial theory. We can verify that this is actually the case by combining these equations to obtain equations of motion for each component field. For example, by using (13) and (14) we eliminate λ and η from the remaining equations. The result is

$$4a\,\partial\!\!\!/\psi-2(2a+1)\gamma_5\partial^\mu\psi_\mu+\gamma_5\partial\!\!\!/\gamma^\mu\psi_\mu+i\Box\chi=0\,, \tag{17}$$
$$2(2a-1)\,\partial\!\!\!/\psi+4\gamma_5\partial^\mu\psi_\mu+\gamma_5\partial\!\!\!/\gamma^\mu\psi_\mu+i\Box\chi=0\,, \tag{18}$$
$$\partial\!\!\!/\psi+\gamma_5\partial^\mu\psi_\mu=0\,. \tag{19}$$

The analysis of these equations shows us that for $a=-2$ they are not independent. On the other hand, for $a\neq-2$ we get

$$\partial\!\!\!/\psi=0\,, \tag{20}$$
$$\partial^\mu\psi_\mu=0\,, \tag{21}$$
$$\Box\chi-i\gamma_5\partial\!\!\!/\gamma^\mu\psi_\mu=0\,. \tag{22}$$

Introducing these results into (13) and (14), one obtains the following equations involving λ and η

$$\partial\!\!\!/\lambda=0\,, \tag{23}$$
$$\Box\chi-2\eta=0\,. \tag{24}$$

We then observe that the equations of motion of the WZ model are obtained among all the equations of the general model. Hence, the WZ model is actually embedded in the Lagrangian (11). However, this compatibility with the WZ model did not permit us to completely fix the relative parameter a. It just says it has to be different from -2. We also observe that it cannot be one because it would rule out the term in FF^* of the Lagrangian (11).

5. Let us now see that the embedding Lagrangian (11) exhibits a kind of gauge symmetry for a specific value of the parameter a. Taking a generic variation of the Lagrangian we obtain

$$\begin{aligned}\delta\mathcal{L}=\delta\bar\psi&\Big[\Big(2a+\frac12\Big)\eta-\Big(a+\frac14\Big)\Box\chi+\frac{i}{2}\partial\!\!\!/\psi+\frac12\gamma_5\partial\!\!\!/\lambda+\frac{i}{2}\gamma_5\partial^\mu\psi_\mu\Big]\\
&+\delta\bar\eta\Big[\Big(2a+\frac12\Big)\psi+\frac{i}{2}\gamma_5\lambda-\frac12\gamma_5\gamma^\mu\psi_\mu+\frac{i}{2}\partial\!\!\!/\chi\Big]\\
&-\delta\bar\lambda\Big[\Big(2a+\frac12\Big)\partial^\mu\psi_\mu+\frac12\gamma_5\partial\!\!\!/\psi-\frac{i}{2}\gamma_5\eta-\frac{i}{2}\partial\!\!\!/\lambda+\frac{i}{4}\gamma_5\Box\chi\Big]\\
&+\delta\bar\psi_\mu\Big[\Big(2a+\frac12\Big)\partial^\mu\lambda-\frac{i}{2}\gamma_5\partial^\mu\psi+\frac12\gamma^\mu\gamma_5\eta+\frac{i}{4}(\gamma^\mu\gamma^\rho\gamma^\nu+\gamma^\nu\gamma^\mu\gamma^\rho)\partial_\nu\psi_\rho+\frac14\gamma_5\gamma^\nu\gamma^\mu\partial\!\!\!/\partial_\nu\chi\Big]\\
&-\delta\bar\chi\Big[\Big(a+\frac14\Big)\Box\psi-\frac{i}{2}\partial\!\!\!/\eta+\frac{i}{4}\gamma_5\Box\lambda-\frac14\gamma_5\gamma^\mu\gamma^\nu\partial\!\!\!/\partial_\mu\psi_\nu\Big]+\delta A^*\Box A+\delta F^*F\,.\end{aligned} \tag{25}$$

Looking at the equation of motion (22), it suggests us that a possible gauge transformation for $\bar\psi_\mu$ should have the form

$$\delta\bar\psi_\mu(x)=\bar\alpha(x)\gamma_\mu\gamma_5\,, \tag{26}$$

where $\alpha(x)$ is a Majorana spinor that plays the role of a gauge parameter. Keeping in mind the mass dimension of the fields that appear in (25), we infer that the gauge transformations for the remaining fields should be

$$
\begin{aligned}
\delta\bar{\psi}(x) &= b\,\bar{\alpha}(x)\,, \\
\delta\bar{\eta}(x) &= c\,\partial_\mu\bar{\alpha}(x)\gamma^\mu\,, \\
\delta\bar{\lambda}(x) &= d\,\bar{\alpha}(x)\gamma_5\,, \\
\delta\bar{\chi}(x) &= e\,\frac{1}{\Box}\partial_\mu\bar{\alpha}(x)\gamma^\mu\,, \\
\delta A^*(x) &= 0\,, \\
\delta F^*(x) &= 0\,,
\end{aligned}
\tag{27}
$$

where b, c, d, and e are parameters to be conveniently fixed. Replacing (26) and (27) into (25) we get that the necessary condition to get the symmetry is

$$
\begin{aligned}
&2b + 2ic + 2id - ie = 8a + 2\,, \\
&2b + (4a+1)2ic + 2id - (4a+1)ie = 2\,, \\
&(4a+1)b + id - ie = 4\,, \\
&(4a+1)b + 2ic + id = -2\,.
\end{aligned}
\tag{28}
$$

These correspond to the coefficients of $\bar{\alpha}\gamma_5\partial\!\!\!/\lambda$, $\bar{\alpha}\partial\!\!\!/\psi$, $\bar{\alpha}\eta$, and $\bar{\alpha}\Box\chi$, respectively. There is still another equation to be verified which is related to the field ψ_μ, namely

$$
\begin{aligned}
&(-6i + 2c + e)\,\bar{\alpha}\gamma_5\gamma^\mu\partial\!\!\!/\psi_\mu \\
&\qquad + [4i + 2bi - 4c - 2d(4a+1)]\,\bar{\alpha}\gamma_5\partial^\mu\psi_\mu = 0\,,
\end{aligned}
\tag{29}
$$

where one cannot infer any conclusion for the coefficients of $\bar{\alpha}\gamma_5\gamma^\mu\partial\!\!\!/\psi_\mu$ and $\bar{\alpha}\gamma_5\partial^\mu\psi_\mu$ because these terms are not independent.

Considering the set given by (28), one can solve it to express b, c, d, and e in terms of a. The result is $b = -1$, $c = 2i$, $d = -i(4a+3)$, and $e = 2i$ (it is important to mention that this solution exists only if $a \neq 0$). Introducing now this result into (29), we get a providential cancelation of the first term. The second one becomes

$$
a(a+1)\,\bar{\alpha}\gamma_5\partial^\mu\psi_\mu = 0\,. \tag{30}
$$

Since a cannot be zero, we get that the symmetry given by (26) and (27) fixes the parameter a into -1 (this value is compatible with all the previous boundary conditions).

6. In this work we have embedded the WZ model in a wider superspace than the one described by chiral and antichiral superfields. We have show that just the fermionic general superfield is appropriated to be used and the consistency condition of the embedding is verified by showing that the same equations of motion of the WZ model are among the equations of motion of the general model. Finally, we have also shown that the embedding theory has a kind of gauge symmetry. This symmetry permit us to fix a relative parameter that appear in the two terms of the Lagrangian.

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APPENDIX A:

In this Appendix, we present the notation, convention and the main identities used throughout the paper. The gamma matrices satisfy the usual relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and $\gamma^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0$. We adopt the metric convention $\eta^{\mu\nu} = \mathrm{diag.}(1, -1, -1, -1)$. We take the completely antisymmetric tensor $\epsilon^{\mu\nu\rho\lambda}$ given by $\epsilon^{0123} = 1$. The matrices γ_5 and $\sigma^{\mu\nu}$ are defined as

$$
\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{A.1}
$$

$$
\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,. \tag{A.2}
$$

Let us list below some useful identities involving gamma matrices

$$
\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho - \eta^{\mu\rho}\gamma^\mu + \eta^{\nu\rho}\gamma^\mu - i\epsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\lambda\,, \tag{A.3}
$$

$$
\gamma_5\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\gamma_5 + \frac{1}{2}\,\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda}\,, \tag{A.4}
$$

$$
\gamma_5\sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda}\,, \tag{A.5}
$$

$$
\gamma^\mu\sigma^{\rho\lambda} = \frac{i}{2}(\eta^{\mu\rho}\gamma^\lambda - \eta^{\mu\lambda}\gamma^\rho) + \frac{1}{2}\epsilon^{\mu\rho\lambda\nu}\gamma_5\gamma_\nu\,, \tag{A.6}
$$

$$
\sigma^{\mu\nu}\gamma^\rho = \frac{i}{2}(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\lambda\,, \tag{A.7}
$$

$$
\begin{aligned}
\sigma^{\mu\nu}\sigma^{\rho\lambda} &= \frac{i}{4}\epsilon^{\mu\nu\rho\lambda}\gamma_5 + \frac{1}{4}(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho}) \\
&\quad - \frac{i}{2}(\eta^{\mu\rho}\eta^{\nu\alpha}\eta^{\lambda\beta} + \eta^{\nu\lambda}\eta^{\mu\alpha}\eta^{\rho\beta} - \rho \leftrightarrow \lambda)\,.
\end{aligned}
\tag{A.8}
$$

Further

$$
\begin{aligned}
\mathrm{tr}\,\gamma^\mu\gamma^\nu &= 4\eta^{\mu\nu}\,, \\
\mathrm{tr}\,\gamma_5 &= 0\,, \\
\mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu &= 0\,, \\
\mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda &= 4i\epsilon^{\mu\nu\rho\lambda}\,, \\
\mathrm{tr}\,\sigma^{\mu\nu}\sigma^{\rho\lambda} &= 4(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho})\,.
\end{aligned}
\tag{A.9}
$$

Considering ψ and χ as Majorana spinors, we also have

$$
\bar{\psi}\chi = \bar{\chi}\psi\,, \tag{A.10}
$$

$$
\bar{\psi}\gamma_5\chi = \bar{\chi}\gamma_5\psi\,, \tag{A.11}
$$

$$
\bar{\psi}\gamma^\mu\gamma_5\chi = \bar{\chi}\gamma^\mu\gamma_5\psi\,, \tag{A.12}
$$

$$
\bar{\psi}\gamma^\mu\chi = -\bar{\chi}\gamma^\mu\psi\,, \tag{A.13}
$$

$$
\bar{\psi}\sigma^{\mu\nu}\chi = -\bar{\chi}\sigma^{\mu\nu}\psi\,, \tag{A.14}
$$

$$
\bar{\psi}\gamma^\mu\gamma^\nu\chi = \bar{\chi}\gamma^\nu\gamma^\mu\psi\,. \tag{A.15}
$$

Using the relations (A.3)-(A.15), we obtain additional relations

$$
\begin{aligned}
\bar\psi\gamma_5\sigma^{\mu\nu}\chi &= -\bar\chi\gamma_5\sigma^{\mu\nu}\psi\,,\\
\bar\psi\gamma_5\gamma^\mu\gamma^\nu\chi &= \bar\chi\gamma_5\gamma^\nu\gamma^\mu\psi\,,\\
\bar\psi\gamma^\mu\gamma^\nu\gamma^\rho\chi &= -\bar\chi\gamma^\rho\gamma^\nu\gamma^\mu\psi\,,\\
\bar\psi\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\chi &= \bar\chi\gamma_5\gamma^\rho\gamma^\nu\gamma^\mu\psi\,,\\
\bar\psi\gamma_5\gamma^\rho\sigma^{\mu\nu}\chi &= -\bar\chi\gamma_5\sigma^{\mu\nu}\gamma^\rho\psi\,,\\
\bar\psi\sigma^{\mu\nu}\gamma^\rho\gamma^\lambda\chi &= \bar\chi\gamma^\lambda\gamma^\rho\sigma^{\mu\nu}\psi\,,\\
\bar\psi\gamma_5\sigma^{\mu\nu}\gamma^\alpha\sigma^{\rho\lambda}\chi &= \bar\chi\gamma_5\sigma^{\rho\lambda}\gamma^\alpha\sigma^{\mu\nu}\psi\,.
\end{aligned} \tag{A.16}
$$

The Fierz identity reads

$$
\frac{1}{4}(\Gamma^A)_{\alpha\beta}(\Gamma_A)_{\sigma\rho} = \delta_{\alpha\rho}\delta_{\beta\sigma}\,, \tag{A.17}
$$

where Γ^A is generically representing the independent matrices: $\Gamma^1 = 1$, Γ^2 to $\Gamma^5 = \gamma^\mu$, $\Gamma^6 = \gamma_5$, Γ^7 to $\Gamma^{10} = \gamma^\mu\gamma_5$, Γ^{11} to $\Gamma^{16} = \sigma^{\mu\nu}$. Concerning to Γ_A, the corresponding relations are almost trivial, we just have to notice the inverse order between γ_5 and γ_μ from Γ_7 to $\Gamma_{10} = \gamma_5\gamma_\mu$. Using the Fierz identity, we obtain

$$
\begin{aligned}
\theta_\alpha\bar\theta_\beta &= -\frac{1}{4}\delta_{\alpha\beta}\bar\theta\theta - \frac{1}{4}\gamma_{5\alpha\beta}\bar\theta\gamma_5\theta\\
&\quad -\frac{1}{4}(\gamma^\mu\gamma_5)_{\alpha\beta}\bar\theta\gamma_5\gamma_\mu\theta\,,\\
\bar\theta\gamma_5\theta\,\bar\theta_\alpha &= -\bar\theta\theta\,(\bar\theta\gamma_5)_\alpha\,,\\
\theta_\alpha\,\bar\theta\gamma_5\theta &= -(\gamma_5\theta)_\alpha\bar\theta\theta\,,\\
\bar\theta\gamma_5\gamma_\mu\theta\,\bar\theta_\alpha &= -\bar\theta\theta\,(\bar\theta\gamma_5\gamma_\mu)_\alpha\,,\\
\theta_\alpha\,\bar\theta\gamma_5\gamma_\mu\theta &= -(\gamma_5\gamma_\mu\theta)_\alpha\bar\theta\theta\,,\\
\bar\theta\gamma_5\theta\,\bar\theta\gamma_5\theta &= -\,(\bar\theta\theta)^2\,,\\
\bar\theta\gamma^\mu\gamma_5\theta\,\bar\theta\gamma^\nu\gamma_5\theta &= \eta^{\mu\nu}(\bar\theta\theta)^2\,,\\
\bar\theta\theta\,\bar\theta\gamma_5\theta &= 0\,,\\
\bar\theta\theta\,\bar\theta\gamma^\mu\gamma_5\theta &= 0\,,\\
\bar\theta\gamma_5\theta\,\bar\theta\gamma^\mu\gamma_5\theta &= 0\,.
\end{aligned} \tag{A.18}
$$

The supersymmetry charge and derivative operators are defined by

$$
\begin{aligned}
Q_\alpha &= \frac{\partial}{\partial\bar\theta_\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu\,,\\
\bar Q_\alpha &= -\frac{\partial}{\partial\theta_\alpha} - i(\bar\theta\gamma^\mu)_\alpha\partial_\mu\,,\\
D_\alpha &= \frac{\partial}{\partial\bar\theta_\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu\,,\\
\bar D_\alpha &= -\frac{\partial}{\partial\theta_\alpha} + i(\bar\theta\gamma^\mu)_\alpha\partial_\mu\,.
\end{aligned} \tag{A.19}
$$

Positive and negative chiralities are defined as

$$
\theta_\pm = \frac{1}{2}(1\pm\gamma_5)\theta\,, \tag{A.20}
$$

consequently,

$$
\begin{aligned}
\frac{\partial}{\partial\theta_\pm}\theta_\pm &= \frac{1}{2}(1\pm\gamma_5)\,,\\
\frac{\partial}{\partial\bar\theta_\pm}\bar\theta_\pm &= \frac{1}{2}(1\mp\gamma_5)\,,\\
\frac{\partial}{\partial\theta_\pm}\bar\theta_\pm &= \frac{1}{2}(1\pm\gamma_5)\gamma^0\,,\\
\frac{\partial}{\partial\bar\theta_\pm}\theta_\pm &= \frac{1}{2}(1\mp\gamma_5)\gamma^0\,,\\
\frac{\partial}{\partial\theta_\pm}\theta_\mp &= 0\,,\\
\frac{\partial}{\partial\bar\theta_\pm}\bar\theta_\mp &= 0\,,\\
\frac{\partial}{\partial\theta_\pm}\bar\theta_\mp &= 0\,,\\
\frac{\partial}{\partial\bar\theta_\pm}\theta_\mp &= 0\,.
\end{aligned} \tag{A.21}
$$

The positive and negative chiral superfields satisfy

$$
D_\mp\Phi_\pm = 0\,. \tag{A.22}
$$

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